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Hamilton–Jacobi modelling of stellar dynamics

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Abstract

One of the physical settings emerging in the galaxy and stellar dynamics is motion of a single star and a stellar cluster about a galaxy center. The potential availability of analytical treatment of this problem stems from the smallness of mass of the star and cluster relative to the galactic mass, giving rise to Hill's restricted three-body problem in the galaxy–cluster–star context. Based on this observation, this paper presents a Hamiltonian approach to modelling stellar motion by the derivation of canonical coordinates for the dynamics of a star relative to a star cluster. First, the Hamiltonian is partitioned into a linear term and a high-order term. The Hamilton–Jacobi equations are solved for the linear part by separation, and new constants for the relative motions are obtained, called epicyclic orbital elements. The effect of an arbitrary cluster potential is incorporated into the analysis by a variation of parameters procedure. A numerical optimization technique is developed based on the new orbital elements, and quasiperiodic stellar orbits are found.

Keywords: Hill's problem; Stellar dynamics; Hamiltonian dynamics

1. Introduction

In the late 19th century, the American mathematician G.W. Hill developed a simple and elegant model for the motion of the Moon around the Earth with the perturbations exerted by the Sun (Hill, 1878). Indeed, to most celestial mechanicians and dynamical astronomers, "Hill's Problem" means a model for planetary motion in which two nearby bodies orbit a much more massive body in quasi-circular orbits. In the current text, however, we use Hill's model to analyze a problem in stellar dynamics.

Consider a star in a star cluster which orbits about a galaxy. The star, cluster and galaxy replace the Moon, Earth, and Sun, respectively, in the classical setting. Although the potentials of the cluster and galaxy are not those of point masses, Hill's problem is a good starting point, which can be easily manipulated via variational techniques to include more complicated effects.

Hill's formulation of the restricted three-body problem (1878) constitutes a ubiquitous approximation for the motion of mutually gravitating bodies about a massive primary. Hill's equations have been used in a variety of celestial mechanics problems and applications (Hénon and Petit, 1986; Namouni, 1999; Villac and Scheeres, 2003; Scheeres et al., 2003). Hénon and Petit (1986) showed that Hill's problem is characterized by the same generality as that of the restricted three-body problem, in the sense that the mass ratio of the orbiting objects can be arbitrary, although still the mass of each body should be very small compared to the massive primary.

Recently, a few works suggested to utilize Hill's equations for modelling the dynamics of a star relative to a star cluster orbiting the galactic center (Heggie, 2000; Fukushige and Heggie, 2000; Ross et al., 1997). Hill's formulation is flexible enough to accommodate extensions to the basic setup, allowing the incorporation of,

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among others, an arbitrary cluster potential and a spherically symmetric galaxy (Binney and Tremaine, 1987).

This paper is aimed at developing canonical orbital elements for modelling the relative stellar motion problem using Hill's equations. In other words, we attempt to find Delaunay-like canonical elements for the relative stellar dynamics.

In order to solve the problem, we first write the Lagrangian for the relative motion, and then perform a Legendre transformation to find the Hamiltonian. We solve the Hamilton–Jacobi equations and treat the gravitational interaction between the star cluster and the star as a perturbation. The new orbital elements, which we termed *epicyclic orbital elements*, are constants of the relative motion, similarly to the elements introduced by Gurfil and Kasdin (2003) for modelling the planetary Hill problem. Due to the fact that the epicyclic orbital elements are canonical, any given external potential can be modelled using Hamilton's equation. This methodology offers a simple and general framework for modelling stellar motion.

2. Equations of motion

The most convenient coordinate system for our problem is the one in which the Hamilton–Jacobi equation most easily separates. Cartesian coordinates turn out to be most convenient. Most of the work in this paper will be confined to a rotating Cartesian Euler–Hill system as shown in Fig. 1. This coordinate system, denoted by $\mathscr{R}$, is defined by the unit vectors $\hat{x},\ \hat{y},\ \hat{z}$. The origin of this coordinate system is set on a circular reference orbit of a star cluster about the galactic center, having radius R. The cluster is rotating about the center with mean motion $n = \sqrt{GM_g/R^3}$, where M_g is the galaxy mass. The reference orbit plane is the fundamental plane, the positive $\hat{x}$-axis points radially outward, the $\hat{y}$-axis is rotated 90° in the direction of motion and lies in the fundamental plane, and the $\hat{z}$-axis completes the setup to yield a Cartesian dextral system.

Using Hill's frame, the equations of motion for a massless star, a spherically symmetric galaxy potential and an arbitrary cluster potential Ψ, the three-dimensional Hill equations modelling the dynamics of a star relative to the cluster are given by (Binney and Tremaine, 1987):

$$\ddot{x} - 2n\dot{y} + (\kappa^2 - 4n^2)x = -\frac{\partial \Psi}{\partial x}, \tag{1}$$

$$\ddot{y} + 2n\dot{x} = -\frac{\partial \Psi}{\partial y}, \tag{2}$$

$$\ddot{z} + n^2 z = -\frac{\partial \Psi}{\partial z}, \tag{3}$$

where κ is termed the *epicyclic frequency*[1] (Binney and Tremaine, 1987). For a point-mass galaxy, $\Psi = -GM_c/r$, $\kappa = n$ and the classical Hill equations are recovered. Based on Eqs. (1)–(3), we can define the *unperturbed* relative motion Lagrangian

$$\mathscr{L}_r^{(0)} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + n(x\dot{y} - y\dot{x} + R\dot{y}) + \frac{3}{2}n^2R^2 - \frac{\kappa^2 - 4n^2}{2}x^2 - \frac{n^2}{2}z^2 - \Psi, \tag{4}$$

and proceed with the analysis using this Lagrangian. Incorporation of perturbations due to mutual gravitation will be included in a perturbing Hamiltonian following a procedure discussed in the next section.

3. The Hamilton–Jacobi solution

In this section, we shall divide the three-degree-of-freedom Hamiltonian of the relative motion into a nominal part and a perturbed part, and then solve the Hamilton–Jacobi equation for the unperturbed, linear system. This solution will provide us with new canonical coordinates and momenta that are constants of the (relative) motion. The perturbation, or variation of parameters equations will then show how these constants vary under the perturbation of mutual gravitation or other perturbations modelled by a perturbing Hamiltonian $\mathscr{H}^{(1)}$.

We therefore aim at casting the Hamiltonian of the relative motion in the form

$$\mathscr{H}_r = \mathscr{H}_r^{(0)} + \mathscr{H}_r^{(1)}$$

and then solve the Hamilton–Jacobi equation for the linear system, represented by the Hamiltonian $\mathscr{H}_r^{(0)}$, without the interaction potential. This solution will yield constant canonical coordinates and momenta. The perturbation equations will then show how these constants vary under the gravitational interaction between the star and the cluster.

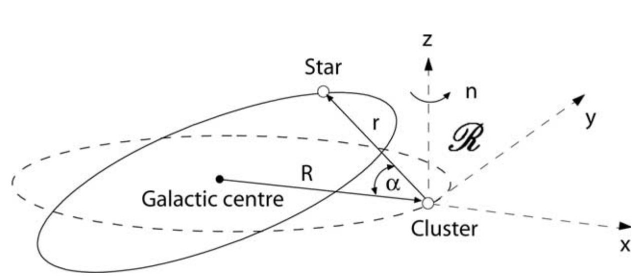

Fig. 1. The Euler–Hill reference frame for modelling relative stellar motion.

[1] The epicyclic frequency describes the timescale for typical non-circular perturbations to oscillate, or grow, in the radial direction of a disc. If we describe the rotation frequency of a star on a nearly circular orbit as $\Omega = v_c/R$, and its radial frequency as the epicyclic frequency κ, then the orbit closes back on itself if Ω/κ is rational.

Finding the Hamiltonian for the system is straightforward. The canonical momenta are found from the usual definition:

$$
\begin{aligned}
p_x &= \frac{\partial \mathscr{L}_r^{(0)}}{\partial \dot{x}} = \dot{x} - ny, \\
p_y &= \frac{\partial \mathscr{L}_r^{(0)}}{\partial \dot{y}} = \dot{y} + n(x+R), \\
p_z &= \frac{\partial \mathscr{L}_r^{(0)}}{\partial \dot{z}} = \dot{z}
\end{aligned} \tag{5}
$$

and then, using the Legendre transformation,

$$\mathscr{H}_r^{(0)} = xp_x + yp_y + zp_z - \mathscr{L}_r^{(0)}, \tag{6}$$

the unperturbed Hamiltonian for relative motion in Cartesian coordinates is found:

$$
\begin{aligned}
\mathscr{H}_r^{(0)} = {}& \frac{1}{2}(p_x + ny)^2 + \frac{1}{2}[p_y - n(x+R)]^2 + \frac{1}{2}p_z^2 \\
& - \frac{3n^2R^2}{2} + \frac{\kappa^2 - 4n^2}{2}x^2 + \frac{n^2}{2}z^2.
\end{aligned} \tag{7}
$$

This Hamiltonian is used to solve the Hamilton–Jacobi equation (Goldstein, 1980). A general explanation of the Hamilton–Jacobi equation is given in Appendix A. Because the Hamiltonian is a constant, Hamilton's principal function easily separates into a time-dependent part summed with Hamilton's characteristic function,

$$S(x, y, z, t) = W(x, y, z) - \alpha_1' t,$$

where α_1' is the constant value of the unperturbed Hamiltonian, $\mathscr{H}_r^{(0)}$. After carrying out a few algebraic manipulations, elaborated in Appendix B, it can be shown that the final generating function, emanating from the solution of the low-order HJ equation, can be written as:

$$S(x, y, z, \alpha_1, \alpha_2, \alpha_3, t) = W(x, y, z) - (\alpha_1 + \alpha_2)t, \tag{8}$$

where

$$
\begin{aligned}
W(x, y, z) = {}& -nxy + \frac{1}{n}\sin^{-1}\left(\frac{nz}{\sqrt{2\alpha_2}}\right)\alpha_2 \\
& + \frac{z\sqrt{-n^2z^2 + 2\alpha_2}}{2} + y(nR + \alpha_3) \\
& + \frac{1}{2\kappa^3}(x\kappa^2 - 2n\alpha_3) \\
& \times \sqrt{-x^2\kappa^2 + 2\alpha_1 + 4nx\alpha_3 - \alpha_3^2} \\
& + \frac{1}{2\kappa^3}\sin^{-1}\left[\frac{-(x\kappa^2) + 2n\alpha_3}{\sqrt{2\kappa^2\alpha_1 + (4n^2 - \kappa^2)\alpha_3^2}}\right] \\
& \times [-2\kappa^2\alpha_1 + (-4n^2 + \kappa^2)\alpha_3^2].
\end{aligned} \tag{9}
$$

Note that we have omitted the constant $3n^2R^2/2$ as it does not affect the solution. It is straightforward to express the new canonical momenta $(\alpha_1, \alpha_2, \alpha_3)$ in terms of the original Cartesian positions and velocities (and thus in terms of the initial conditions). For instance, α_3 is given by Eq. (75) using Eq. (5). Eq. (71) is used to find α_2, substituting p_z from Eq. (5) for $\mathrm{d}W_3/\mathrm{d}z$. Finally, $\alpha_1 = \alpha_1' - \alpha_2 + 3n^2R^2/2$ is simply the value of the Hamiltonian and is thus given by Eq. (7) with the momenta substituted from Eq. (5). The result is:

$$\alpha_1 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{\kappa^2 - 4n^2}{2}x^2, \tag{10}$$

$$\alpha_2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}n^2z^2, \tag{11}$$

$$\alpha_3 = \dot{y} + 2nx. \tag{12}$$

The canonical coordinates (Q_1, Q_2, Q_3) and the corresponding phase variables $(\beta_1, \beta_2, \beta_3)$ are found via the partial derivatives of the generating functions in Eq. (8) with respect to each of the new canonical momenta,

$$
\begin{aligned}
Q_i &= \frac{\partial[S(x, y, z, \alpha_1, \alpha_2, \alpha_3, t) + \alpha_1' t]}{\partial \alpha_i} \\
&= \frac{\partial[W(x, y, z, \alpha_1, \alpha_2, \alpha_3)]}{\partial \alpha_i}
\end{aligned} \tag{13}
$$

yielding

$$Q_1 = t + \beta_1 = \frac{1}{\kappa}\tan^{-1}\left(\frac{\kappa^2x - 2n\dot{y} - 4n^2x}{\kappa\dot{x}}\right), \tag{14}$$

$$Q_2 = t + \beta_2 = \frac{1}{n}\tan^{-1}\left(\frac{nz}{\dot{z}}\right), \tag{15}$$

$$
\begin{aligned}
Q_3 = \beta_3 = {}& \frac{(-\kappa^2\dot{y} + 4\dot{y}n^2 - 2n\kappa^2x + 8n^3x)}{\kappa^3} \\
& \times \tan^{-1}\left(\frac{-\kappa^2x + 2n\dot{y} + 4n^2x}{\kappa\dot{x}}\right) - \frac{2n\dot{x}\kappa - y\kappa^3}{\kappa^3}.
\end{aligned} \tag{16}
$$

Solving Eqs. (10)–(16) for x, y, and z yields the *generating solution* for the Cartesian relative position components in terms of the new constants of the motion, the canonical momenta $(\alpha_1, \alpha_2, \alpha_3)$ and the canonical coordinates (Q_1, Q_2, Q_3):

$$x(t) = 2n\alpha_3 + \sqrt{2\kappa^2\alpha_1 - (\kappa^2 - 4n^2)\alpha_3^2}\sin(\kappa Q_1), \tag{17}$$

$$
\begin{aligned}
y(t) = {}& \kappa^3 Q_3 + \alpha_3\kappa(\kappa^2 - 4n^2)Q_1 \\
& + 2n\sqrt{2\kappa^2\alpha_1 + (\kappa^2 - 4n^2)\alpha_3^2}\cos(\kappa Q_1),
\end{aligned} \tag{18}
$$

$$z(t) = \frac{1}{n}\sqrt{2\alpha_2}\sin(nQ_2). \tag{19}$$

From Eqs. (10)–(16) (or, alternatively, by differentiating Eqs. (17)–(19) with respect to time) we can also obtain the expressions for the Cartesian relative velocity components in terms of α_1, α_2, α_3 and Q_1, Q_2, Q_3:

$$\dot{x}(t) = \kappa\sqrt{2\kappa^2\alpha_1 - (\kappa^2 - 4n^2)\alpha_3^2}\cos(\kappa Q_1), \tag{20}$$

$$
\begin{aligned}
\dot{y}(t) = {}& \alpha_3\kappa(\kappa^2 - 4n^2) \\
& - 2n\kappa\sqrt{2\kappa^2\alpha_1 + (\kappa^2 - 4n^2)\alpha_3^2}\sin(\kappa Q_1),
\end{aligned} \tag{21}
$$

$$\dot{z}(t) = \sqrt{2\alpha_2}\cos(nQ_2). \tag{22}$$

We call the new constants of the motion $\Xi = [\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3]$ *epicyclic orbital elements* for the relative stellar motion. They are defined on the manifold $\mathscr{O} \times \mathbb{S}^3$, where $\mathscr{O} = \mathbb{R} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R} \subset \mathbb{R}^3$.

While these canonical variables can be used in the final equations of motion, one more modification dramatically simplifies the final result. We define a new momentum variable,

$$\alpha'_1 = \frac{1}{2}\left(-1 + \frac{4n^2}{\kappa^2}\right)\alpha_3^2 + \alpha_1$$

and solve for the new low-order Hamiltonian

$$H^{(0)} = \alpha_2 + \frac{1}{2}\left(1 - \frac{4n^2}{\kappa^2}\right)\alpha_3^2 + \alpha'_1 \tag{23}$$

and the new generating function

$$\begin{aligned} W_{\text{new}} = & -nxy + \frac{1}{n}\sin^{-1}\left(\frac{nz}{\sqrt{2\alpha_2}}\right)\alpha_2 + \frac{z\sqrt{-n^2z^2 + 2\alpha}}{2} \\ & + y(nR + \alpha_3) + \frac{1}{2\kappa^2}(x\kappa^2 - 2n\alpha_3) \\ & \times \sqrt{-(x^2\kappa^2) + 2\alpha'_1 + 4nx\alpha_3 - \alpha_3^2} \\ & + \frac{1}{2\kappa^3}\sin^{-1}\left[\frac{-x\kappa^2 + 2n\alpha_3}{\sqrt{2\kappa^2\alpha'_1 + (4n^2 - \kappa^2)\alpha_3^2}}\right] \\ & \times [-2\kappa^2\alpha'_1 + (-4n^2 + \kappa^2)\alpha_3^2]. \end{aligned} \tag{24}$$

By modifying the generating function accordingly, we obtain equations for the new canonical momenta and coordinates in terms of the Cartesian variables:

$$\alpha'_1 = \frac{x^2(-4n^2 + \kappa^2)}{2} + \frac{\dot{x}^2}{2} + \frac{\dot{y}^2}{2} + \frac{1}{2}\left(-1 + \frac{4n^2}{\kappa^2}\right)(2nx + \dot{y})^2, \tag{25}$$

$$\alpha'_2 = \frac{1}{2}\dot{z}^2 + \frac{1}{2}n^2z^2, \tag{26}$$

$$\alpha'_3 = \dot{y} + 2nx, \tag{27}$$

$$Q'_1 = t + \beta_1 = \frac{1}{\kappa}\tan^{-1}\left[\frac{x(-4n^2 + \kappa^2) - 2n\dot{y}}{\kappa\dot{x}}\right], \tag{28}$$

$$Q'_2 = t + \beta_2 = \frac{1}{n}\tan^{-1}\left(\frac{nz}{\dot{z}}\right), \tag{29}$$

$$Q'_3 = t\left(1 - \frac{4n^2}{\kappa^2}\right)\alpha_3 + \beta_3 = y - \frac{2n\dot{x}}{\kappa^2}. \tag{30}$$

From this point onward in the paper, these new canonical elements will be used and the primes will be dropped for convenience.

Solving Eqs. (25)–(30) for x, y, and z yields the generating solution for the Cartesian relative position components in terms of the new constants of the motion:

$$x(t) = \frac{2n\alpha_3 + \sqrt{2}\kappa\sin(\kappa Q_1)\sqrt{\alpha_1}}{\kappa^2}, \tag{31}$$

$$y(t) = Q_3 + \frac{2\sqrt{2}n\cos(\kappa Q_1)\sqrt{\alpha_1}}{\kappa^2}, \tag{32}$$

$$z(t) = \frac{\sqrt{2}\sin(nQ_2)\sqrt{\alpha_2}}{n}. \tag{33}$$

Thus, as is well known from solutions of Hill's unperturbed equations, the motion consists of a periodic out-of-plane oscillation parameterized by α_2, Q_2, a periodic in-plane motion described by α_1, Q_1, and the center of the x–y ellipse is parameterized by α_3 and Q_3. The y-invariance is given by the shift Q_3. Non-constant values of α_3 and Q_3 represent the secular drift of an orbit, i.e., an escaping star (Heggie, 2000). From Eqs. (25)–(30) (or, alternatively, by differentiating Eqs. (31)–(33) with respect to time) we can also obtain the expressions for the Cartesian relative velocity components in terms of $(\alpha_1, \alpha_2, \alpha_3)$ and (Q_1, Q_2, Q_3):

$$\dot{x}(t) = \sqrt{2}\cos(\kappa Q_1)\sqrt{\alpha_1}, \tag{34}$$

$$\dot{y}(t) = \left(1 - \frac{4n^2}{\kappa^2}\right)\alpha_3 - \frac{2\sqrt{2}n\sin(\kappa Q_1)\sqrt{\alpha_1}}{\kappa}, \tag{35}$$

$$\dot{z}(t) = \sqrt{2}\cos(nQ_2)\sqrt{\alpha_2}. \tag{36}$$

Finally, it is often convenient to have expressions for the original canonical momenta in terms of the new elements. These are found from the transformation equations:

$$p_x = \frac{-(n\kappa^2Q_3) + \sqrt{2}(-2n^2 + \kappa^2)\cos(\kappa Q_1)\sqrt{\alpha_1}}{\kappa^2}, \tag{37}$$

$$p_y = nR + \left(1 - \frac{2n^2}{\kappa^2}\right)\alpha_3 - \frac{\sqrt{2}n\sin(\kappa Q_1)\sqrt{\alpha_1}}{\kappa}, \tag{38}$$

$$p_z = \sqrt{2}\cos(nQ_2)\sqrt{\alpha_2}. \tag{39}$$

4. Modified epicyclic elements

The epicyclic elements described above provide a convenient parametrization of a first-order relative motion orbit in terms of amplitude and phase. However, the variational equations for these elements, accounting for the gravitational interaction, can become quite complicated and numerically sensitive. This is a concern when some of the amplitudes approach zero, resulting in phase terms which are ill-defined. For these situations it is convenient to introduce an alternative set of constants in terms of amplitude variables only. We call these *modified epicyclic orbital elements*, $\Xi' = [a_1, a_2, a_3, b_1, b_2, b_3]$, and define them via the contact transformation:

$$a_1 = \frac{\sqrt{2}\sin(\kappa\beta_1)\sqrt{\alpha_1}}{\sqrt{\kappa}}, \tag{40}$$

$$b_1 = \frac{\sqrt{2}\cos(\kappa\beta_1)\sqrt{\alpha_1}}{\sqrt{\kappa}}, \tag{41}$$

$$a_2 = \frac{\sqrt{2}\cos(n\beta_2)\sqrt{\alpha_2}}{\sqrt{n}}, \tag{42}$$

$$b_2 = \frac{\sqrt{2}\sin(n\beta_2)\sqrt{\alpha_2}}{\sqrt{n}}, \tag{43}$$

$$a_3 = \alpha_3, \tag{44}$$

$$b_3 = Q_3. \tag{45}$$

It can be easily shown that the transformation in Eqs. (40)–(45) is symplectic. That is, letting $M = \partial\Xi'/\partial\Xi$, it is straightforward to show that:

$$MJM^T = J, \tag{46}$$

where J is the orthoskew matrix $[0, I; -I, 0]$. Thus, the new elements are also canonical and satisfy Hamilton's equations. In some cases, the variational equations for these elements will be much easier to work with. In terms of the new canonical momenta (a_1, a_2, a_3) and new canonical coordinates (b_1, b_2, b_3), the generating solution becomes:

$$x(t) = \frac{\cos(t\kappa)a_1}{\sqrt{\kappa}} + \frac{2na_3}{\kappa^2} + \frac{\sin(t\kappa)b_1}{\sqrt{\kappa}}, \tag{47}$$

$$y(t) = \frac{-2n\sin(t\kappa)a_1}{\kappa^{\frac{3}{2}}} + \frac{2n\cos(t\kappa)b_1}{\kappa^{\frac{3}{2}}} + b_3, \tag{48}$$

$$z(t) = \frac{\cos(nt)a_2}{\sqrt{n}} + \frac{\sin(nt)b_2}{\sqrt{n}} \tag{49}$$

for the new Hamiltonian:

$$H^{(0)} = \frac{1}{2}\left(1 - \frac{4n^2}{\kappa^2}\right)a_3^2. \tag{50}$$

Variation of the parameters Ξ or Ξ' due to perturbations such as a gravitational interaction can be obtained via Hamilton's equations on a perturbing Hamiltonian, and the resulting time varying parameters $\Xi(t)$ or $\Xi'(t)$ can then be substituted into the generating solutions to yield the exact relative motion description in the configuration space $\mathbb{R}^3$. This is the subject of the next section.

5. Gravitational interaction analysis via variation of parameters

The primary value of the canonical approach is the ease with which equations for the variations of parameters can be found. For example, the variations of the epicyclic orbital elements are given by Hamilton's equations on the perturbation Hamiltonian, $\mathscr{H}_r^{(1)}$:

$$\dot{b}_i = \frac{\partial\mathscr{H}_r^{(1)}}{\partial a_i} = \frac{\partial\mathscr{H}_r^{(1)}}{\partial x}\frac{\partial x}{\partial a_i} + \frac{\partial\mathscr{H}_r^{(1)}}{\partial y}\frac{\partial y}{\partial a_i} + \frac{\partial\mathscr{H}_r^{(1)}}{\partial z}\frac{\partial z}{\partial a_i}, \tag{51}$$

$$\dot{a}_i = -\frac{\partial\mathscr{H}_r^{(1)}}{\partial b_i} = -\left(\frac{\partial\mathscr{H}_r^{(1)}}{\partial x}\frac{\partial x}{\partial b_i} + \frac{\partial\mathscr{H}_r^{(1)}}{\partial y}\frac{\partial y}{\partial b_i} + \frac{\partial\mathscr{H}_r^{(1)}}{\partial z}\frac{\partial z}{\partial b_i}\right). \tag{52}$$

These can be used to find the effect on the elements of any number of perturbations which are derived from a conservative potential, such as the gravitational interaction between the star and the cluster. Moreover, the developed framework supports general models for the cluster potential, such as King's model (King, 1962). For a point-mass galaxy (setting $\mathscr{H}_r^{(1)} = \Psi = -GM_c/r$ and utilizing the normalization $\kappa = n = 1$), Eqs. (51) and (52) yield

$$\dot{a}_1 = \frac{-2ny\mu\cos(t\kappa)}{r^3\kappa^{\frac{3}{2}}} - \frac{x\mu\sin(t\kappa)}{r^3\sqrt{\kappa}}, \tag{53}$$

$$\dot{b}_1 = \frac{x\mu\cos(t\kappa)}{r^3\sqrt{\kappa}} - \frac{2ny\mu\sin(t\kappa)}{r^3\kappa^{\frac{3}{2}}}, \tag{54}$$

$$\dot{a}_2 = -\left(\frac{z\mu\sin(nt)}{\sqrt{n}r^3}\right), \tag{55}$$

$$\dot{b}_2 = \frac{z\mu\cos(nt)}{\sqrt{n}r^3}, \tag{56}$$

$$\dot{a}_3 = -\frac{y\mu}{r^3}, \tag{57}$$

$$\dot{b}_3 = a_3\left(1 - \frac{4n^2}{\kappa^2}\right) + \frac{2nx\mu}{r^3\kappa^2}, \tag{58}$$

where $x = x(\Xi', t)$, $y = y(\Xi', t)$, $z = z(\Xi', t)$, $r = r(\Xi', t)$ are functions of the modified epicyclic elements. Eqs. (53)–(58) constitute canonical Hill equations. They can be re-written in the state-space form

$$\Sigma : \dot{\Xi}'(t) = F(\Xi', t). \tag{59}$$

These equations can be studied analytically or numerically in order to detect periodic and quasiperiodic orbits for the relative stellar dynamics. This is the subject of the next section.

6. Numerical search for bounded relative orbits

We illustrate the use of the newly defined orbital elements by implementing a numerical search procedure (Gurfil and Kasdin, 2003) aimed at detecting bounded planar solutions for the dynamical system Σ. To this end, we express the modified epicyclic elements using Fourier series expansions of the form

$$a_1(t) = a_0 + \sum_{k=1}^{\infty} a_k^s\sin(kt) + a_k^c\cos(kt), \tag{60}$$

$$a_3(t) = b_0 + \sum_{k=1}^{\infty} b_k^s\sin(kt) + b_k^c\cos(kt), \tag{61}$$

$$b_1(t) = c_0 + \sum_{k=1}^{\infty} c_k^s\sin(kt) + c_k^c\cos(kt), \tag{62}$$

$$b_3(t) = d_0 + \sum_{k=1}^{\infty} d_k^s\sin(kt) + d_k^c\cos(kt) \tag{63}$$

and perform a numerical search using a gradient-based parameter optimization routine, comprising the following steps. First, we substitute Eq. (60), truncated at some given l, into Eqs. (53)–(58). The unknown parameters are:

$$\mathbf{a} = \{a_0, a_1^s, \ldots a_l^s, a_1^c, \ldots a_l^c, b_0, b_1^s, \ldots b_l^s, b_1^c, \ldots b_l^c, c_0, c_1^s, \ldots c_l^s, b_1^c, \ldots c_l^c d_0, d_1^s, \ldots d_l^s, d_1^c, \ldots d_l^c\}. \tag{64}$$

Next, we find the optimal Fourier series coefficients by performing the unconstrained least-squares minimization

$$\mathbf{a}^* = \arg\min_{\mathbf{a}} \int_0^{t_f} \|\dot{\Xi}'(\mathbf{a}, t) - F([\Xi'(\mathbf{a}, t)])\|^2\,\mathrm{d}t. \tag{65}$$

After the optimization routine converges to a (possibly local) minimum, we calculate the initial conditions from:

$$a_1(0) = a_0 + \sum_{k=1}^{l} a_k^c, \tag{66}$$

$$a_3(0) = b_0 + \sum_{k=1}^{l} b_k^c, \tag{67}$$

$$b_1(0) = c_0 + \sum_{k=1}^{l} c_k^c, \tag{68}$$

$$b_3(0) = d_0 + \sum_{k=0}^{l} d_k^c \tag{69}$$

and use them to integrate the differential equations (59). If the minimum found is larger than zero, there will be differences between the time histories of the epicyclic elements as obtained from the simulation, and Eq. (60)–(63). However, after running a sufficient number of random initial guesses, there is a large ensemble of solutions yielding a minimum which is sufficiently close to zero, implying that the integrated solutions and the static solutions match.

We emphasize that the optimization procedure described above is *static*. That is, the differential equations (53)–(58) are transformed into algebraic equations using the pre-defined, periodic, topology of the (candidate) solutions given by Eqs. (60)–(63).

To illustrate the results obtained using the described methodology, we chose randomly selected initial guesses for **a***. A number of retrograde quasiperiodic orbits were found. Fig. 2 depicts some of these orbits. The top plots in this figure describe the orbits in the x–y plane, i.e., the configuration space, and the bottom plots describe the orbits in terms of the guiding center, defined by $(2a_3, b_3)$. All the axes are normalized by Hill's parameter, $\epsilon = (\mu/3)^{1/3}$. The motion is coorbital if the guiding center is contained within the annulus $\pm\epsilon$. In this case, the dynamics are determined by the 1:1 mean motion commensurability.

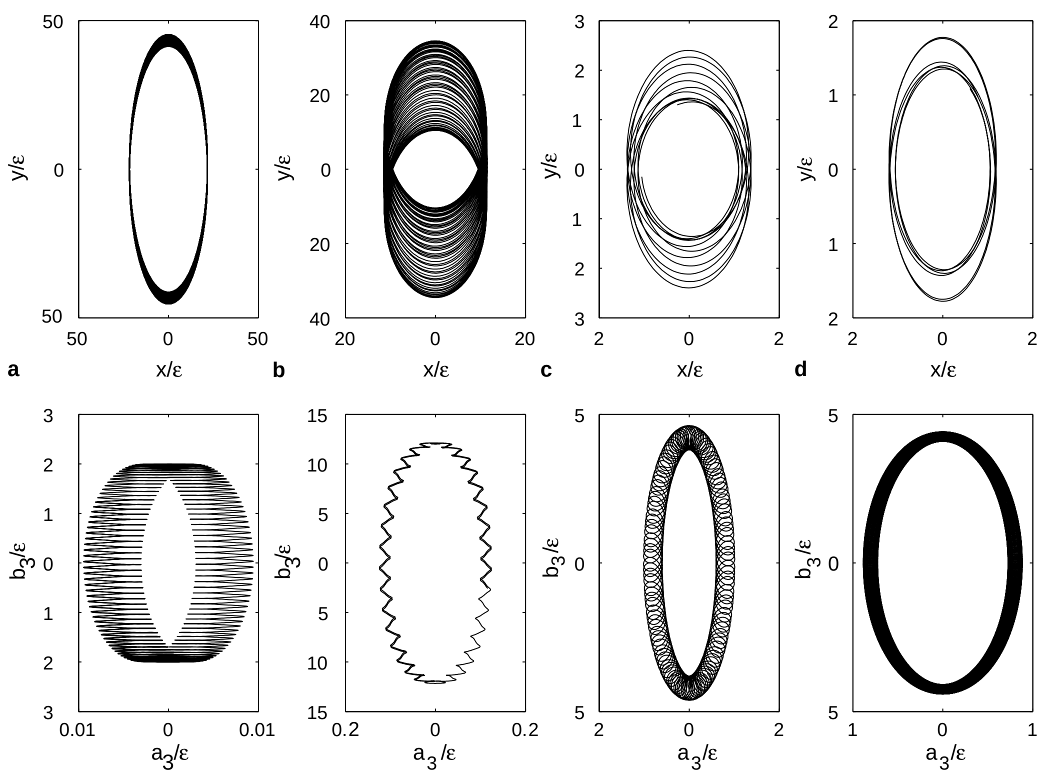

Fig. 2. Quasiperiodic star orbits in the x–y plane (top plots) and their guiding centers in the b_3–a_3 plane (bottom plots).

Table 1
Initial conditions of modified epicyclic elements

Element	Orbit a	Orbit b	Orbit c	Orbit d
a_1	9.878828344993e − 3	−1.685487979534e − 2	−1.310080444760e − 3	1.942441419644e − 3
a_3	2.170959044953e − 6	−6.917608836309e − 6	5.247175473481e − 4	−7.997868241214e − 4
b_1	3.084348228288e − 2	1.944695949659e − 4	3.102346029393e − 3	2.234208495246e − 3
b_3	−2.829622548335e − 3	−1.780556876619e − 2	−4.910831767320e − 3	−4.130667313510e − 3

Orbit (a) represents motion with small magnitude of the guiding center (meaning small variation of the normalized coordinate a_3/ϵ), orbit (b) represents motion with medium magnitude, and orbits (c) and (d) represent motion with large magnitude of the guiding center, which is still contained within the annulus of the Lagrangian equilibrium points. Equivalent results where obtained by Namouni (1999) for the celestial dynamics case using classical orbital elements. The initial conditions used to integrate the equations of motion (59) are given in Table 1. We emphasize that by selecting suitable initial conditions, the center of mass of the star–cluster system will follow the reference unit circle. The analysis of the relative motion is then carried out relative to the known motion of the center of mass.

7. Conclusions

This paper developed a Hamiltonian framework for the analysis of stellar motion in terms of canonical relative motion elements termed "epicyclic" orbital elements. The epicyclic elements are constants of the motion.

The underlying model was a Hill-like three-body problem model, which also contained an epicyclic frequency. This approach permits a convenient analysis of the gravitational interaction between the star and the star cluster via a variation-of-parameters procedure. The resulting canonical elements were transformed into new orbital elements, termed "modified" epicyclic elements, via a contact transformation, yielding a compact form of the variational equations.

This unique approach renders an important analytical insight into the relative stellar motion problem, and is very useful for numerical probing of the state-space dynamics. Some characteristic orbits were found using a numerical optimization routine, which was applied on the newly found orbital elements.

There are many extensions to this approach. We are currently working on developing a solution for motion about elliptical orbits and the incorporation of King's model for the interaction potential.

Appendix A. The Hamilton–Jacobi equation

The Hamilton–Jacobi equation is a methodology for solving integrable dynamics problems via canonical transformations. We briefly describe the derivation here utilizing Goldstein (1980). Given a set of generalized coordinates $(q, \dot{q})$ and a Lagrangian $\mathscr{L}$, the canonical momenta are found via:

$$p_i = \frac{\partial \mathscr{L}}{\partial q_i}.$$

The Hamiltonian is then given by the Legendre transformation:

$$\mathscr{H}(q,p,t) = \dot{q}_i p_i - \mathscr{L}(q,\dot{q},t).$$

The n second order equations of motion for the problem can then be alternatively written as $2n$ first order equations for $\dot{q}$ and $\dot{p}$ (Hamilton's equations):

$$\dot{q}_i = \frac{\partial \mathscr{H}}{\partial p_i},$$
$$\dot{p}_i = -\frac{\partial \mathscr{H}}{\partial q_i}.$$

These canonical coordinates are not unique. If we consider a transformation of the phase space to a new set of coordinates $Q_i(q,p,t)$ and $P_i(q,p,t)$, we can ask for the class of transformations for which the new coordinates also satisfy Hamilton's equations on the new Hamiltonian $\mathscr{K}(Q,P,t)$. Such a transformation is called canonical. A common approach to the transformation is via generating functions. For some function $F_2(q,P,t)$, a transformation is canonical provided that:

$$p_i = \frac{\partial F_2}{\partial q_i},$$
$$Q_i = \frac{\partial F_2}{\partial P_i},$$
$$\mathscr{K} = \mathscr{H} + \frac{\partial F_2}{\partial t}.$$

The Hamilton–Jacobi problem comes from asking for the special canonical transformations for which $\mathscr{K} \equiv 0$ and thus the new canonical coordinates are constants of the motion. If such a transformation can be found, the equations of motion have been solved. The generating function for such a transformation is called Hamilton's principle function, $S(q,\alpha,t)$, where α_i is the new constant canonical momentum, P_i. This generating function can be found by setting the expression for $\mathscr{K}$ equal to zero while substituting from the transformation equations:

$$\mathscr{H}\left(q_1,\ldots,q_n;\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n};t\right) + \frac{\partial S}{\partial t} = 0$$

This is known as the Hamilton–Jacobi equation for S. Note that in the special case where $\mathscr{H}$ is independent of time it is a constant of the motion and can be set equal to α_1. In this case, the HJ equation separates and we write Hamilton's principle function in terms of $W(q_i,\alpha_i)$, called Hamilton's characteristic function, and time:

$$S(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n;t) = W(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n) - \alpha_1 t$$

The Hamilton–Jacobi equation for W then reduces to:

$$\mathscr{H}\left(q_1,\ldots,q_n;\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right) = \alpha_1.$$

Appendix B. Solution of the Hamilton–Jacobi equation

The Hamilton–Jacobi equation reduces to a partial differential equation in $W(x,y,z)$:

$$\alpha_1' = \frac{1}{2}\left(\frac{\partial W}{\partial x} + ny\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial y} - n(x+R)\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial z}\right)^2 - \frac{3n^2R^2}{2} + \frac{\kappa^2 - 4n^2}{2}x^2 + \frac{n^2}{2}z^2. \quad (70)$$

Not unexpectedly, the z-coordinate immediately separates. Writing the characteristic function as $W(x,y,z) = W'(x,y) + W_3(z)$, the HJ equation separates into:

$$\alpha_2 = \frac{1}{2}\left(\frac{\mathrm{d}W_3}{\mathrm{d}z}\right)^2 + \frac{n^2}{2}z^2, \quad (71)$$

$$\alpha_1' + \frac{3n^2R^2}{2} - \alpha_2 = \frac{1}{2}\left(\frac{\partial W'}{\partial x} + ny\right)^2 + \frac{1}{2}\left[\frac{\partial W'}{\partial y} - n(x+R)\right]^2 + \frac{\kappa^2 - 4n^2}{2}x^2. \quad (72)$$

Eq. (71) is just the HJ equation for simple harmonic motion. It is easily solved via quadrature:

$$W_3(z) = \int \sqrt{2\alpha_2 - n^2z^2}\,\mathrm{d}z = \frac{1}{2}\left[z\sqrt{2\alpha_2 - n^2z^2} + \frac{2\alpha_2}{n}\sin^{-1}\left(\frac{nz}{\sqrt{2\alpha_2}}\right)\right]. \quad (73)$$

The solution of Eq. (72) is more subtle. We separate by using the well-known constant of integration of Hill's equations, stemming from the integral of Eq. (2) with $\Psi = 0$:

$$\int \ddot{y} + 2n\dot{x} = \text{const.} \quad (74)$$

Setting α_3 equal to the constant of Eq. (74) and putting it in terms of the canonical momentum, the third integration constant of the HJ equation becomes:

$$\alpha_3 = p_y + n(x - R). \quad (75)$$

Using the fact that $p_y = \partial W'/\partial y$, the remaining HJ equation (Eq. (72)) separates if we let

$$W'(x,y) = W_1(x) + W_2(y) - nyx \quad (76)$$

so that

$$\frac{\mathrm{d}W_2}{\mathrm{d}y} = \alpha_3 + nR$$

and thus $W_2 = (\alpha_3 + nR)y$ by quadrature. Eq. (72) then simplifies to:

$$\left(\frac{\mathrm{d}W_1}{\mathrm{d}x}\right)^2 + \kappa^2x^2 - 4\alpha_3 nx = 2\alpha_1 - \alpha_3^2, \quad (77)$$

where we have used $\alpha_1 = \alpha_1' - \alpha_2 + 3n^2R^2/2$. This equation is again easily integrated for W_1 by quadrature,

$$W_1 = \int \sqrt{2\alpha_1 - \alpha_3^2 + 4\alpha_3 nx - \kappa^2x^2}\,\mathrm{d}x \quad (78)$$

yielding

$$W_1 = \frac{1}{2\kappa^2}(x\kappa^2 - 2n\alpha_3) \times \sqrt{-x^2\kappa^2 + 2\alpha_1 + 4nx\alpha_3 - \alpha_3^2} + \frac{1}{2\kappa^3}\sin^{-1}\left[\frac{-x\kappa^2 + 2n\alpha_3}{\sqrt{2\kappa^2\alpha_1 - (-4n^2 + \kappa^2)\alpha_3^2}}\right] \times [-2\kappa^2\alpha_1 + (-4n^2 + \kappa^2)\alpha_3^2]. \quad (79)$$

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